

Self-gravitating Bose-Einstein condensates and the Thomas-Fermi approximation

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Self-gravitating Bose-Einstein condensates have been proposed in various astrophysical contexts, including Bose-stars and BEC dark matter halos. These systems are described by a combination of the Gross-Pitaevskii and Poisson equations (the GPP system). In the analysis of these hypothetical objects, the Thomas-Fermi (TF) approximation is widely used. This approximation is based on the assumption that in the presence of a large number of particles, the kinetic term in the Gross-Pitaevskii energy functional can be neglected, yet this assumption is violated near the condensate surface. We also show that the total energy of the self-gravitating condensate in the TF-approximation is positive. The stability of a self-gravitating system is dependent on the total energy being negative. Therefore, the TF approximation is ill suited to formulate initial conditions in numerical simulations. As an alternative, we offer an approximate solution of the full GPP system.

Self-gravitating Bose-Einstein condensates can undergo gravitational collapse [1] and may form gravitationally stable structure [2]. have been proposed in various astrophysical contexts, including Bose-stars [3–5] and BEC dark matter halos [6–8]. A self-gravitating Bose-Einstein condensate is described by a combination of the Gross-Pitaevskii equation and Poisson's equation for gravity. In units such that the BEC particle mass is $m = 1$ and also $\hbar = 1$, the time-independent Gross-Pitaevskii equation (GPE) can be written in an attractively simple form:

$$-\frac{1}{2}\nabla^2\Psi + (V + c|\Psi|^2 - \mu)\Psi = 0, \quad (1)$$

where Ψ is the BEC wavefunction, V is the gravitational potential, c is the BEC coupling coefficient and μ is the chemical potential, the presence of which guarantees the conservation of energy. We normalize the wavefunction such that the number of particles is $\int_V |\Psi|^2 = N$.

The energy functional from which with the GPE (1) can be derived using the variational principle (cf. [9, 10]; note the additional factor of $1/2$ in front of V , required to avoid double counting the gravitational potential energy between two regions of the condensate as V is itself a function of $|\Psi|^2$) is given by

$$\mathcal{E} = \frac{1}{2}|\nabla\Psi|^2 + \left(\frac{1}{2}V - \mu\right)|\Psi|^2 + \frac{1}{2}c|\Psi|^4. \quad (2)$$

In the Thomas-Fermi (TF) approximation [4, 5], kinetic energy is neglected. Therefore, the time-independent GPE takes on the following simplified form:

$$(V + c|\Psi|^2 - \mu)\Psi \simeq 0. \quad (3)$$

If V is not dependent on Ψ , this equation can be solved directly for $|\Psi|^2$:

$$|\Psi|^2 \simeq \frac{\mu - V}{c}. \quad (4)$$

Moreover, if we require the wavefunction to vanish at infinity, we must have

$$\mu - V \rightarrow 0 \quad (5)$$

at infinity.

If V is dependent on Ψ , the situation becomes somewhat more complicated. In particular, in the GPP system, the relationship between V and Ψ is given by Poisson's equation:

$$\nabla^2 V = 4\pi G|\Psi|^2, \quad (6)$$

where G is the gravitational constant. Solving the GPE (1) for V in the TF approximation,

$$V \simeq \mu - c|\Psi|^2, \quad (7)$$

and substituting this solution back into Poisson's equation (6), we get

$$\nabla^2(\mu - c|\Psi|^2) \simeq 4\pi G|\Psi|^2. \quad (8)$$

If $\mu = \text{const.}$, we are left with

$$\left[\nabla^2 + \frac{4\pi G}{c}\right]|\Psi|^2 \simeq 0, \quad (9)$$

which is an homogeneous Helmholtz-type equation for $|\Psi|^2$, spherically symmetric solutions of which are

$$|\Psi|^2 \simeq C_1 \frac{\sin kr}{r} + C_2 \frac{\cos kr}{r}, \quad (10)$$

where $k^2 = 4\pi G/c$, while C_1 and C_2 are integration constants. To avoid solutions that are singular at the origin $r = 0$, we must set $C_2 = 0$. On the other hand, $\sin kr/r$ (and thus, $|\Psi|^2$) vanishes at $r = \pi/k$. Therefore, we set $r_0 = \pi/k$ as the radius of the condensate. This determines C_1 since we require that

$$\int_V C_1 \frac{\sin kr}{r} dV = N. \quad (11)$$

This integral can be readily evaluated:

$$\int_V C_1 \frac{\sin kr}{r} dV = 4\pi C_1 \int_0^{r_0} r \sin \frac{\pi r}{r_0} dr = 4C_1 r_0^2, \quad (12)$$

hence $C_1 = N/4r_0^2$. Therefore, the TF approximation for the GPP is given by the Lane-Emden type solution

$$|\Psi|^2 = \frac{N}{4r_0^2} \frac{\sin(\pi r/r_0)}{r}, \quad (13)$$

for $0 \leq r \leq r_0 = \sqrt{\pi c/4G}$.

Given $|\Psi|^2$, we can solve the GPE (1) for V :

$$V = \mu - c|\Psi|^2 = \mu - \frac{cN}{4r_0^2} \frac{\sin(\pi r/r_0)}{r}, \quad (14)$$

again for $0 \leq r \leq r_0$.

At r_0 and beyond, the condensate vanishes, and the gravitational potential becomes that of a point mass M (where $M = Nm$ is the total mass of the condensate), i.e., $V = -GN/r$ ($r_0 \leq r$). At the boundary, these two forms must agree. This can be achieved by setting

$$\mu = -\frac{GN}{r_0}. \quad (15)$$

This clarifies the role of the chemical potential in the case of the GPP system in the TF approximation: its presence ensures that the gravitational potential takes on the standard form outside the condensate and vanishes at infinity.

The energy density of the time-independent GPE in the Thomas-Fermi limit is given by

$$\mathcal{E} \simeq \left(\frac{1}{2}V - \mu\right) |\Psi|^2 + \frac{1}{2}c|\Psi|^4, \quad (16)$$

or, after substituting the solution for V from the GPE (1),

$$\mathcal{E} \simeq -c|\Psi|^4 + \frac{1}{2}c|\Psi|^4 = -\frac{1}{2}c|\Psi|^4. \quad (17)$$

$$\mathcal{E} \simeq -\frac{1}{2}\mu|\Psi|^2 = \frac{GN}{r_0}|\Psi|^2. \quad (18)$$

To find the total energy, we integrate over the condensate volume:

$$\begin{aligned} E &= \int_V \mathcal{E} dV = 4\pi \int_0^{r_0} r^2 \mathcal{E} dr \\ &\simeq \frac{\pi GN^2}{r_0^3} \int_0^{r_0} r \sin(\pi r/r_0) dr = \frac{GN^2}{r_0}, \end{aligned} \quad (19)$$

or, after restoring units,

$$E \simeq \frac{Gm^2 N^2}{r_0}. \quad (20)$$

The positive sign of the total energy implies that the solution for a self-gravitating BEC using the TF approximation is inherently unstable.

This result is based on the assumption that the kinetic energy can be neglected. Now that we have an explicit solution for $|\Psi|^2$, this assumption can be verified by direct substitution into the energy functional (2). When we do so we find that, using the solution given by Eq. (13), the condensate kinetic energy,

$$\begin{aligned} \text{KE} &= \int \frac{1}{2} |\nabla \Psi|^2 dV = 4\pi \int_0^{r_0} \frac{r^2}{2} |\nabla \Psi|^2 dr \\ &= \frac{\pi N}{2r_0^2} \int_0^{r_0} r^2 \left| \nabla \sqrt{\frac{\sin(\pi r/r_0)}{r}} \right|^2 dr \end{aligned} \quad (21)$$

is divergent for any $r_0 > 0$.

The implication of this divergence is that the assumption behind the TF approximation, namely that the kinetic term in the GPE (1) can be neglected, is maximally violated in the case of the self-gravitational GPP system.

While this divergent behavior of the Thomas-Fermi approximation is known [10], it has especially important implications for numerical simulations of self-gravitating Bose-Einstein condensates that use this approximation to model the initial state (see, e.g., [11, 12]). The neglected kinetic energy term can become arbitrarily large near the surface of the condensate. The magnitude of the kinetic term is dependent on nonphysical simulation parameters, such as the numerical integration step size or even small rounding errors. An ill-defined or outright divergent kinetic energy term in an initial configuration yields unpredictable behavior in the simulation. Furthermore, even if we ignore the issue of numerical stability, the positive total energy is especially troublesome in the case of a self-gravitating system: the stability of such a system is dependent on $E < 0$.

This finding agrees with the author's experience using numerical simulation code [11] that was designed to model the GPP system. Initial versions of the code ran differently in different programming environments (e.g., single vs. double precision, FORTRAN vs. C), processors (Intel x86 vs. GPGPU) and operating systems (Linux vs. Windows). This is clearly not permissible: the results, apart from accuracy and rounding issues, should not be dependent on such factors. Indeed, the present study arose as a result of systematically analyzing the failure of these algorithms to produce consistent results.

Numerically stable simulations require an initial state that is not dependent on nonphysical parameters and does not lead to a divergent energy term. Therefore, we now aim to find an approximate solution of the GPP system in the spherically symmetric case without resorting to the TF approximation. Assuming spherical coordinates and a spherically symmetric condensate, the GPE (1) becomes an equation of a single independent variable r . Let us denote $\partial/\partial r = \partial_r$ for brevity, while noting the form of the Laplacian in spherical coordinates, $\nabla^2 = \partial_r^2 + (2/r)\partial_r$. We can then write the GPE (1) as

$$\partial_r^2 \Psi + \frac{2}{r} \partial_r \Psi - 2V\Psi - 2c|\Psi|^2\Psi + 2\mu\Psi = 0, \quad (22)$$

whereas Poisson's equation (6) becomes:

$$\partial_r^2 V + \frac{2}{r} \partial_r V = 4\pi G|\Psi|^2. \quad (23)$$

Let us now consider writing the wavefunction as

$$\Psi(r) = |\Psi|(r)e^{i\phi(r)}, \quad (24)$$

in which case

$$d\Psi = e^{i\phi} [d|\Psi| + i|\Psi|d\phi], \quad (25)$$

$$d^2\Psi = e^{i\phi} \{d^2|\Psi| - |\Psi|(d\phi)^2 + i[2d|\Psi|d\phi + |\Psi|d^2\phi]\}, \quad (26)$$

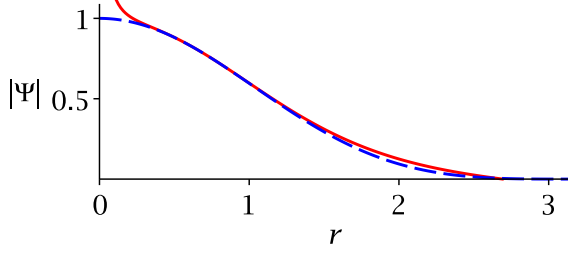


FIG. 1: Numerical solution (solid red line) of Eq. (33), giving the norm of the wavefunction $|\Psi|$ of a spherically symmetric self-gravitating BEC as a function of radius r , compared to the approximation (dashed blue line) provided in Eq. (35).

and the GPE (1) can be rewritten, after dividing through by $e^{i\phi}$, as

$$\begin{aligned} & \partial_r^2 \Psi - |\Psi|(\partial_r \phi)^2 + \frac{2}{r} \partial_r |\Psi| \\ & + i \left(2\partial_r |\Psi| \partial_r \phi + |\Psi| \partial_r^2 \phi + \frac{2}{r} |\Psi| \partial_r \phi \right) \\ & - 2V|\Psi| - 2c|\Psi|^3 + 2\mu|\Psi| = 0. \end{aligned} \quad (27)$$

Since $|\Psi|$, ϕ , V , c and μ are all real, the real and imaginary parts of this equation can be separated:

$$\partial_r^2 |\Psi| + \frac{2}{r} \partial_r |\Psi| - \left[(\partial_r \phi)^2 + 2V + 2c|\Psi|^2 - 2\mu \right] |\Psi| = 0, \quad (28)$$

$$\partial_r^2 \phi + 2 \left[\partial_r \ln |\Psi| + \frac{1}{r} \right] \partial_r \phi = 0. \quad (29)$$

Equation (29) can be integrated:

$$\partial_r \phi = \frac{C}{r^2 |\Psi|^2}, \quad (30)$$

where C is an integration constant with the dimensions of $r|\Psi|^2$. It seems that $C = 0$ is not only a valid choice but the only choice that does not result in ϕ becoming singular at the origin (assuming the wavefunction does not vanish at the origin). Therefore, $\phi = \text{const}$.

Under these circumstances, the GPE (1) will read

$$\partial_r^2 |\Psi| + \frac{2}{r} \partial_r |\Psi| - [2V + 2c|\Psi|^2 - 2\mu] |\Psi| = 0, \quad (31)$$

which is readily solvable for V algebraically:

$$V = \frac{\partial_r^2 |\Psi|}{2|\Psi|} + \frac{\partial_r |\Psi|}{r|\Psi|} - c|\Psi|^2 + \mu. \quad (32)$$

This result can be substituted back into Poisson's equation, yielding a fourth-order ordinary differential equation in $|\Psi|$:

$$\begin{aligned} & \partial_r^4 |\Psi| - \frac{2\partial_r |\Psi| \partial_r^3 |\Psi|}{|\Psi|} + \frac{4\partial_r^3 |\Psi|}{r} - \frac{(\partial_r^2 |\Psi|)^2}{|\Psi|} - 4c|\Psi|^2 \partial_r^2 |\Psi| \\ & + \frac{2(\partial_r |\Psi|)^2 \partial_r^2 |\Psi|}{|\Psi|^2} - \frac{8\partial_r |\Psi| \partial_r^2 |\Psi|}{r|\Psi|} + \frac{4(\partial_r |\Psi|)^3}{r|\Psi|^2} \\ & - 4c|\Psi|(\partial_r |\Psi|)^2 - \frac{8c|\Psi|^2 \partial_r |\Psi|}{r} - 8\pi G|\Psi|^3 = 0. \end{aligned} \quad (33)$$

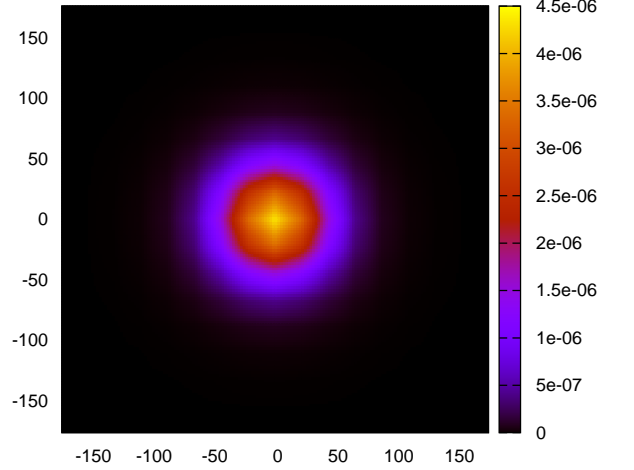


FIG. 2: Density cross section of a stable simulated $1 M_\odot$, $r \simeq 50$ km Bose-star (or stellar core) after approximately 300,000 numerical iterations that corresponds to 3 seconds [13]. (For comparison, the period of a circular orbit at $r = 50$ km is approximately 0.006 s.) Axes are in km, density is in units of 10^{30} kg/km³ $\simeq 0.5 M_\odot/\text{km}^3$.

This equation can be solved numerically. Given that it is a fourth-order homogeneous differential equation in $|\Psi|$, it has a very large solution space, parameterized by boundary or initial conditions, such as the values of $|\Psi|$ and its first three derivatives at some value of r . A hint for a suitable solution comes from numerical simulation [11], where we find that apparently stable nonrotating spherically symmetric solutions converge on $|\Psi|^2 \propto [\sin(r/r_0)/(r/r_0)]^{2\alpha}$, with $3 \lesssim \alpha \lesssim 4$. This approximate solution has many desirable properties. It is smooth in the interval $0 \geq r/r_0 \geq \pi$. The corresponding kinetic energy (21) is finite in the same interval. Moreover, it is possible to compute the condensate mass, which is given by

$$M(r) = \int_V |\Psi|^2 dV = 4\pi \int_0^r r'^2 |\Psi(r')|^2 dr', \quad (34)$$

and this, too, is finite and well-behaved. Therefore, we find that the following initial approximation for the magnitude of the BEC-Poisson wavefunction:

$$|\Psi| \propto \left| \frac{\sin r/r_0}{r/r_0} \right|^3, \quad (35)$$

agrees well with a numerical solution (except for very small values of r), as shown in Fig. 1. Furthermore, this choice yields a corresponding solution of Eq. (6) for the gravitational potential that is finite, negative, and vanishes at infinity, as expected. The stability of these solutions is confirmed by numerical simulation of a condensate using Eq. (35) as an initial approximation. An example result is shown in Fig. 2; more results and analysis will be reported elsewhere [13] as they become available.

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